JOURNAL OF APPROXIMATION THEORY 74, 237–248 (1993)

Interpolation by Generalized Polynomials with Restricted Ranges

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Communicted by Frank Deutsch

Received March 8, 1990; accepted in revised form May 8, 1992

In 1990, A. Horwitz proved a theorem about interpolation by restricted range polynomials and asked some ralted questions. This paper gives affirmative answers to Horwitz' questions and generalizes his theorem.

INTRODUCTION

In 1980, Briggs and Rubel [1] proved the existence of a non-negative polynomial of degree $\leqslant n$ that interpolates a non-negative continuous function at $n+1$ distinct points. In other words, they showed that for some choice of $n+1$ distinct points, the unique Lagrange interpolant to f at those points is a non-negative polynomial. Recently, applying a perturbation method, Horwitz [2] gave a similar result for interpolation by polynomials with restricted ranges, that is,

THEOREM H. *Suppose* $f \in C[0, 1]$ *with* $0 \leqslant f(x) \leqslant 1$, $x \in [0, 1]$. *Let n be a positive integer and assume that*

(i) *n is even if* $f(0) = f(1) = 0$ *or* 1,

(ii) *n is odd if* $f(0) = 0$ *and* $f(1) = 1$, *or* $f(0) = 1$ *and* $f(1) = 0$.

Then there exist a polynomial p of degree $\leqslant n$ *with* $0 \leqslant p(x) \leqslant 1$, $x \in [0, 1]$, *which interpolates f at $n+1$ distinct points in* $[0, 1]$.

In addition, Horwitz asked (see [2]): Can assumptions (i) and (ii) be removed in Theorem H? Does Theorem H hold when the upper and lower functions are not necessarily constant? And does it hold for Chebyshev systems other than the polynomials? Unfortunately, as Horwitz pointed out, the techniques in [2] do not seem to answer these questions.

In this paper, using some perturbation techniques different from Horwitz' we give affirmative answers to the above questions.

0021-9045/93 $5.00

MAIN RESULT

Let $[a, b]$ be a finite interval, n a positive integer. For linearly independent functions $\varphi_0, ..., \varphi_n \in C[a, b]$, we say that $\Phi_n = \text{span}\{\varphi_0, ..., \varphi_n\}$ is the set of all the generalized polynomials. Given two functions $l(x)$ and $u(x)$ defined on $[a, b]$, by

$$K(l, u) = \{p \in \Phi_n : l(x) \leqslant p(x) \leqslant u(x), x \in [a, b]\}$$

we denote the subset of the generalized polynomials having restricted ranges.

According to the notion introduced in [3], we call $\{\varphi_0, ..., \varphi_n\}$ an extended Chebyshev system of order 2 on $[a, b]$ provided that each φ_j, $j = 0, ..., n$ has a continuous derivative, and for arbitrary $a \leqslant x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b$, where no group of three consecutive x_i's can take the same value, it follows that

$$\begin{vmatrix} \tilde{\varphi}_0(x_0) & \tilde{\varphi}_1(x_0) & \cdots & \tilde{\varphi}_n(x_0) \\ \tilde{\varphi}_0(x_1) & \tilde{\varphi}_1(x_1) & \cdots & \tilde{\varphi}_n(x_1) \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{\varphi}_0(x_n) & \tilde{\varphi}_1(x_n) & \cdots & \tilde{\varphi}_n(x_n) \end{vmatrix} > 0,$$

where $\tilde{\varphi}_j(x_i) = \varphi_j(x_i)$ if $x_{i-1} < x_i$, $\tilde{\varphi}_j(x_i) = \varphi_j'(x_i)$ if $x_{i-1} = x_i$, $0 \leqslant j \leqslant n$. For $p \in \Phi_n$, where $\{\varphi_0, ..., \varphi_n\}$ is an extended Chebyshev system of order 2, we say that x is a zero of order 2 of p if $p(x) = p'(x) = 0$. Then p has at most n zeros in $[a, b]$ counting multiplicities up to 2. And when p has n distinct zeros, $p(x)$ changes sign as x passes through each of its zeros and preserves the same sign between two consecutive zeros.

THEOREM. *Let $\{\varphi_0, ..., \varphi_n\}$ be an extended Chebyshev system of order 2 on $[a, b]$, and $p_1, p_{-1} \in \Phi_n$ be subject to $p_1(x) < p_{-1}(x)$, $x \in [a, b]$. If $f \in C[a, b]$ satisfies $p_1(x) \leqslant f(x) \leqslant p_{-1}(x)$, $x \in [a, b]$, then there exists a $p \in K(p_1, p_{-1})$ which interpolates f at $n+1$ distinct points in $[a, b]$.*

Proof. We can assume that $f \notin \Phi_n$ and

$$d = \tfrac{1}{2} \inf\{|\xi' - \xi''| : \xi', \xi'' \in D, \xi' \neq \xi''\} > 0,$$

where

$$D = \{x \in [a, b] : f(x) = p_1(x) \text{ or } p_{-1}(x)\},$$

for otherwise $p = f$ or p_δ ($\delta = 1$ or -1) satisfies the requirements of the theorem.

Based on Theorem 3.1 in [4], there exists a generalized polynomial p^* which is the best uniform approximation (with the uniform norm $\|\cdot\| = \sup_{x\in[a,b]} |\cdot|$) to f from $K(p_1, p_{-1})$. Let

$$p_\delta^* = p^* - p_\delta, \qquad \delta = \pm 1;$$
$$C_\delta = \{x \in [a, b] : p_\delta^*(x) = 0\}, \qquad \delta = \pm 1;$$
$$E_\delta = \{x \in [a, b] : f(x) - p^*(x) = \delta \|f - p^*\|\}, \qquad \delta = \pm 1;$$
$$\sigma(x) = \begin{cases} 1, & \text{if } x \in E_1 \cup C_1, \\ -1, & \text{if } x \in E_{-1} \cup C_{-1}; \end{cases}$$

and

$$C_\pm = C_1 \cup C_{-1}.$$

Based on Theorem 3.2 in [4], we can find $n+2$ points $x_1 < \cdots < x_{n+2}$ such that $x_i \in C_\pm \cup E_1 \cup E_{-1}$ and

$$\sigma(x_i) = (-1)^{i+1} \sigma(x_1), \qquad i = 2, ..., n+2.$$

Write

$$X = \{x_i\}_{i=1}^{n+2}$$

and

$$C = \{\xi \in C_\pm : p^*(\xi) = f(\xi)\},$$
$$c = \tfrac{1}{3} \min\{|\xi' - \xi''| : \xi', \xi'' \in C_\pm \cup \{a, b\}, \xi' \neq \xi''\}.$$

Let $|C|$ and $|C_\pm|$ denote the numbers of elements of C and $C_\pm$, respectively. If $|C| \geqslant n+1$, then p^* is the required polynomial. And if $|C_\pm| = 0$, then by the alternating property p^* still meets the requirement of the theorem. So we assume that

$$|C| \leqslant n, \qquad |C_\pm| \geqslant 1.$$

Moreover, it can be proved that

$$|C_\pm| \leqslant n+1.$$

In fact, if there exist at least $n+2$ points in $C_\pm$, then we can find a $\delta \in \{1, -1\}$ with $|C_\delta| \geqslant (n+3)/2$ if n is odd. Since each point in C_δ is a zero of p_δ^* of order 2 with the exception of at most two endpoints, p_δ^* has at least $n+1$ zeros counting multiplicities, which is a contradiction. When n is even, if there exists a $\delta \in \{1, -1\}$ such that $|C_\delta| \geqslant (n+2)/2 + 1$, then p_δ^* has at least $n+2$ zeros, and if $|C_1| = |C_{-1}| = (n+2)/2$, then there exists a $\delta \in \{1, -1\}$ for which at most one endpoint belongs to C_δ, and hence p_δ^* has at least $n+1$ zeros. Again, these are impossible.

In what follows, we find n points $\{y_i\}_{i=1}^n$, a $q \in \Phi_n$ with n zeros $\{y_i\}$, and an $\varepsilon > 0$ such that $p^* - \varepsilon q$ meets the requirements of the theorem.

Case 1. If $|C_\pm| \leqslant n$, write $n' = |C_\pm|$ and

$$C_\pm = \{\xi_1, ..., \xi_{n'-1}, \xi_n\},$$

where

$$\xi_i < \xi_j, \qquad i < j. \tag{1}$$

Let

$$J = \{1, ..., n'-1, n\}.$$

Choose arbitrarily $n - n'$ points in $(\xi_{n'-1} + c, \xi_n - c)$ (or in $(a, \xi_n - c)$ if $n' = 1$),

$$y_{n'} < \cdots < y_{n-1},$$

and set

$$\sigma = 1.$$

Case 2. If $|C_\pm| = n+1$, then we can find an $\xi^* \in C_\pm \backslash C$ since $|C| \leqslant n$. Write

$$C_\pm \backslash \{\xi^*\} = \{\xi_1, ..., \xi_n\}$$

subject to (1). Let

$$J = \{1, ..., n\}.$$

Choose a $\xi_{j^*} \in C_\pm$ adjacent to ξ^*, and let

$$\sigma = \begin{cases} (-1)^{j^*-1} \sigma(\xi_{j^*}), & \sigma(\xi^*)\, \sigma(\xi_{j^*})(\xi_{j^*} - \xi^*) > 0, \\ (-1)^{j^*} \sigma(\xi_{j^*}), & \sigma(\xi^*)\, \sigma(\xi_{j^*})(\xi_{j^*} - \xi^*) < 0. \end{cases} \tag{2}$$

In both cases, we set a y_j adjacent to ξ_j for each $j \in J$. Let

$$\sigma_j = (-1)^{j-1} \sigma \sigma(\xi_j), \qquad j \in J. \tag{3}$$

Write

$$X_1 = \{\xi_j : j \in J,\ p^{*\prime}_{\sigma(\xi_j)}(\xi_j) \neq 0\},$$

$$X_2 = \{\xi_j : j \in J,\ \xi_j + \sigma_j c \notin [a, b]\}$$

and

$$X_* = X_1 \cup X_2.$$

Clearly

$$X_* \subset \{a, b\}. \tag{4}$$

Since $X \cap C \subset \{\xi_j : j \in J\}$ we can write

$$(X \cap C) \backslash X_* = \{\xi_{j_1}, ..., \xi_{j_m}\} \tag{5}$$

with $j_1 < \cdots < j_m$, $j_i \in J$. Now, for $j \in J \backslash \{j_i\}_{i=1}^m$, let

$$y_j = \begin{cases} \xi_j, & \text{if} \quad \xi_j \in X_*, \\ \xi_j + \sigma_j c, & \text{otherwise;} \end{cases} \tag{6}$$

and for $i = 1, ..., m$, let

$$y_{j_i} \in Y_{j_i} \tag{7}$$

be undetermined, where

$$Y_{j_i} = (\xi_{j_i}, \xi_{j_i} + \sigma_{j_i} \rho) \qquad \text{or} \qquad (\xi_{j_i} + \sigma_{j_i} \rho, \xi_{j_i}) \tag{8}$$

with

$$\rho \leqslant \min\{c, d\} \tag{9}$$

a positive constant, which we determine later.

Now, take a non-vanishing $q \in \Phi_n$ having n zeros $\{y_j\}_{j=1}^n$. For each $\xi_j \notin X_*$, $j \in J$, based on (6), (7), and (8) we see that y_j is on the right side of ξ_j if $\sigma_j = 1$ and on the left side if $\sigma_j = -1$. Therefore, in Case 1 we can find from (3) that for any pair of consecutive points $\xi' < \xi''$ in $C_\pm \backslash X_*$

$$\begin{array}{l} \text{there exist an even number of zeros of } q \text{ in } (\xi', \xi''), \\ \qquad \text{if} \quad \sigma(\xi')\,\sigma(\xi'') > 0 \\ \text{there exist an odd number of zeros of } q \text{ in } (\xi', \xi''), \\ \qquad \text{if} \quad \sigma(\xi')\,\sigma(\xi'') < 0. \end{array} \tag{$*$}$$

Moreover, if ξ' or $\xi'' \in X_2 \subset X_*$, then by the definition of X_2, $(*)$ is still true though (6) holds. So $(*)$ holds for $C_\pm \backslash X_1$. In Case 2, the situation is similar if $\xi^* \notin \{\xi', \xi''\}$. And if $\xi^* < \xi_{j^*}$, then by (2) and (3) we have

$$\sigma_{j^*} = \begin{cases} 1, & \sigma(\xi^*)\,\sigma(\xi_{j^*}) > 0, \\ -1, & \sigma(\xi^*)\,\sigma(\xi_{j^*}) < 0. \end{cases}$$

Considering in addition that there exists no zero of q "adjacent to" ξ^*, we can see that $(*)$ holds if $\xi' = \xi^*$, $\xi'' = \xi_{j^*}$, and if $j^* > 1$, $\xi' = \xi_{j^*-1}$, $\xi'' = \xi^*$ then $(*)$ holds as well. Similarly, when $\xi^* > \xi_{j^*}$ the same conclusion holds.

Therefore, multiplied by -1 if necessary, q satisfies

$$\sigma(\xi)\, q(\xi) \leqslant 0, \qquad \xi \in C_\pm \backslash X_1$$

and

$$\sigma(\xi)\, q(x) \leqslant 0, \qquad x \in \bar{O}(\xi, \rho), \quad \xi \in (C_\pm \backslash X_1) \backslash \{\xi_{j_i}\}_{i=1}^m, \tag{10}$$

where $\bar{O}(\xi, \rho)$ denotes the closure of $O(\xi, \rho)$, the ρ-neighbourhood of ξ.

Now suppose that we have determined ρ subject to (9) and y_j $(j \in \{j_i\}_{i=1}^m)$ subject to (7) (and hence q) and have found an $\varepsilon > 0$ such that

$$\|\varepsilon q\| < \min\{e_1, e_2, e_3\}, \tag{11}$$

where

$$\begin{cases} e_1 = \min\limits_{x \in [a,b] \backslash O(C_\pm, \rho)} \{\min\limits_{\delta = \pm 1} |p_\delta^*(x)|\}, \\ e_2 = \min\limits_{\xi \in C_\pm} \{\min\limits_{x \in \bar{O}(\xi, \rho)} |p^*_{-\sigma(\xi)}(x)|\}, \end{cases} \tag{12}$$

$$e_3 = \min_{x \in X \backslash C} |f(x) - p^*(x)|; \tag{13}$$

and

$$|\varepsilon q(x)| \leqslant |p^*_{\sigma(\xi)}(x)|, \qquad x \in O(\xi, \rho), \quad \xi \in X_1 \tag{14}$$

if $X_1 \neq \Phi$; moreover

$$\min_{x \in \bar{O}(\xi_{j_i}, \rho)} \sigma(\xi_{j_i})[p^*_{\sigma(\xi_{j_i})}(x) - \varepsilon q(x)] = 0, \qquad i = 1, ..., m. \tag{15}$$

Then by (11), (12), (10), and (14) we see that

$$p_1(x) \leqslant p^*(x) - \varepsilon q(x) \leqslant p_{-1}(x), \qquad x \in [a, b] \Big\backslash \bigcup_{i=1}^m O(\xi_{j_i}, \rho).$$

Combining this with (15) we get

$$p^* - \varepsilon q \in K(p_1, p_{-1}).$$

On the other hand, by (5) we can rewrite

$$X = (X \backslash C) \cup \{\xi_{j_1}, ..., \xi_{j_m}\} \cup (X \cap C \cap X_*),$$

and, based on (15), for each ξ_{j_i} we can choose an adjacent point $x'_{j_i} \in \bar{O}(\xi_{j_i}, \rho)$ satisfying

$$\sigma(\xi_{j_i})[p^*_{\sigma(\xi_{j_i})}(x'_{j_i}) - \varepsilon q(x'_{j_i})] = 0, \qquad i = 1, ..., m.$$

It is then easy to check that when x takes the values of $(X \setminus C) \cup \{x'_{j_i}\}_{i=1}^m$ one by one in order of magnitude, $f(x) - [p^*(x) - \varepsilon q(x)]$ takes positive and negative values alternately, because for the points in $X \setminus C$ we have (11) and (13), and for x'_{j_i} we have $p_1(x'_{j_i}) < f(x'_{j_i}) < p_{-1}(x'_{j_i})$, which is obtained from (9) and the definition of d. In addition, the function equals zero when $x \in X \cap C \cap X_*$ by (6). So from (4) we conclude that $p^* - \varepsilon q$ interpolates f at $n+1$ distinct points.

It remains to find ρ, ε, and y_{j_i}, $i = 1, ..., m$ (and hence q), satisfying the demands mentioned above. Let

$$Q(x, \eta_1, ..., \eta_m) = (-1)^{j_1} \sigma_{j_1} \sigma(\xi_{j_1}) \begin{vmatrix} \varphi_0(x) & \cdots & \varphi_n(x) \\ \varphi_0(y_1) & \cdots & \varphi_n(y_1) \\ \cdots & \cdots & \cdots \\ \varphi_0(y_{j_1-1}) & \cdots & \varphi_n(y_{j_1-1}) \\ \varphi_0(\eta_1) & \cdots & \varphi_n(\eta_1) \\ \varphi_0(y_{j_1+1}) & \cdots & \varphi_n(y_{j_1+1}) \\ \cdots & \cdots & \cdots \\ \varphi_0(y_{j_m-1}) & \cdots & \varphi_n(y_{j_m-1}) \\ \varphi_0(\eta_m) & \cdots & \varphi_n(\eta_m) \\ \varphi_0(y_{j_m+1}) & \cdots & \varphi_n(y_{j_m+1}) \\ \cdots & \cdots & \cdots \\ \varphi_0(y_n) & \cdots & \varphi_n(y_n) \end{vmatrix}. \qquad (16)$$

By Q_0 and Q_i we denote $(\partial/\partial x)\, Q$ and $(\partial/\partial \eta_i)\, Q$, respectively. From the definition of an extended Chebyshev system we have

$$(-1)^{j_i + j_1} \sigma_{j_1} \sigma(\xi_{j_1})\, Q_0(\xi_{j_i}, \xi_{j_1}, ..., \xi_{j_m}) > 0, \qquad i = 1, ..., m.$$

Then for each $i = 1, ..., m$, by the continuity of Q_0 there exists $\rho_i > 0$ such that

$$(-1)^{j_i + j_1} \sigma_{j_1} \sigma(\xi_{j_1})\, Q_0(x, \eta_1, ..., \eta_m) > 0,$$
$$x \in \bar{O}(\xi_{j_i}, \rho_i), \quad \eta_\nu \in \bar{O}(\xi_{j_\nu}, \rho_i), \quad \nu = 1, ..., m. \qquad (17)$$

Furthermore, considering the fact that $Q_i(\xi_{j_k}, \xi_{j_1}, \ldots, \xi_{j_m}) = 0$, $k \neq i$ $(k, i = 1, \ldots, m)$, we can find an $\alpha > 0$ and a positive

$$\rho' \leqslant \min_{i=1,\ldots,m} \rho_i$$

such that for each $i = 1, \ldots, m$

$$|Q_i(x, \eta_1, \ldots, \eta_m)| > \alpha, \qquad x \in \bar{O}(\xi_{j_i}, \rho'), \quad \eta_v \in \bar{O}(\xi_{j_v}, \rho'), \quad v = 1, \ldots, m, \tag{18}$$

and

$$|Q_i(x, \eta_1, \ldots, \eta_m)| \leqslant \frac{\alpha}{m}, \qquad x \in \bar{O}(\xi_{j_k}, \rho') \quad \text{with} \quad k \neq i, \quad \eta_v \in \bar{O}(\xi_{j_v}, \rho'), \quad v = 1, \ldots, m. \tag{19}$$

Now let

$$\mu = \max_{\substack{\eta_i \in \bar{O}(\xi_{j_i}, \rho') \\ i=1,\ldots,m}} \{ \max_{x \in [a, b]} |Q(x, \eta_1, \ldots, \eta_m)| \}.$$

If $X_1 = \varnothing$, let

$$\rho = \min\{\rho', c, d\},$$

and

$$\varepsilon = \frac{1}{2} \frac{\min\{e_1, e_2, e_3\}}{\mu}.$$

Then for

$$q(x) = Q(x, y_{j_i}, \ldots, y_{j_m}), \tag{20}$$

where $y_{j_i} \in Y_{j_i}$ (see (8)) is undetermined, and (11) holds. And if $X_1 \neq \varnothing$, we set

$$\mu' = \min_{\xi \in X_1} |p^{*\prime}_{\sigma(\xi)}(\xi)|$$

and

$$\mu'' = \max_{\substack{\eta_i \in \bar{O}(\xi_{j_i}, \rho'') \\ i=1,\ldots,m}} \{ \max_{\substack{x \in \bar{O}(\xi, \rho'') \\ \xi \in X_1}} |Q_0(x, \eta_1, \ldots, \eta_m)| \},$$

where $0 < \rho'' \leqslant \rho'$ satisfies

$$|p^{*\prime}_{\sigma(\xi)}(x)| > \frac{\mu'}{2}, \qquad x \in O(\xi, \rho''), \qquad \xi \in X_1.$$

Let

$$\rho = \min\{\rho'', c, d\},$$

and

$$\varepsilon = \frac{1}{2} \min \left\{ \frac{\min\{e_1, e_2, e_3\}}{\mu}, \frac{\mu'}{\mu''} \right\}.$$

Then clearly q in (20) satisfies (11). And in addition we have (14) because

$$\varepsilon q'(x) \leqslant \frac{\mu'}{2}, \qquad x \in O(\xi, \rho), \qquad \xi \in X_1.$$

We must choose $y_{j_i} \in Y_{j_i}$ so that (15) holds. Fix $(\eta_2, ..., \eta_m) \in \bar{Y}_{j_2} \times \cdots \times \bar{Y}_{j_m}$ arbitrarily. Let

$$M(\eta_1) = \min_{x \in \bar{O}(\xi_{j_1}, \rho)} \sigma(\xi_{j_1})[p^*_{\sigma(\xi_{j_1})}(x) - \varepsilon Q(x, \eta_1, ..., \eta_m)].$$

Let us consider the sign of $M(\eta_1)$. For $p^*_{\sigma(\xi_{j_1})}$ clearly we have

$$p^*_{\sigma(\xi_{j_1})}(\xi_{j_1}) = 0,$$
$$p^{*\prime}_{\sigma(\xi_{j_1})}(\xi_{j_1}) = 0.$$

As for Q, we have

$$Q(\xi_{j_1}, \xi_{j_1}, \eta_2, ..., \eta_m) = 0,$$

and by (17)

$$\sigma_{j_1} \sigma(\xi_{j_1})\, Q_0(\xi_{j_1}, \xi_{j_1}, \eta_2, ..., \eta_m) > 0,$$

and hence

$$\sigma(\xi_{j_1})\, Q(x, \xi_{j_1}, \eta_2, ..., \eta_m) > 0, \qquad x \in Y_{j_1}.$$

Thus we see that

$$M(\xi_{j_1}) < 0.$$

On the other hand, since $Q(x, \xi_{j_1}+\sigma_{j_1}\rho, \eta_2, ..., \eta_m)$ preserves the same signs for various $x \in O(\xi_{j_1}, \rho)$ and equals zero if $x=\xi_{j_1}+\sigma_{j_1}\rho$, by (17) (let $i=1$, $x=\eta_1=\xi_{j_1}+\sigma_{j_1}\rho$) it is easy to check that

$$Q(\xi_{j_1})\, Q(x, \xi_{j_1}+\sigma_{j_1}\rho, \eta_2, ..., \eta_m)<0, \qquad x \in O(\xi_{j_1}, \rho).$$

Considering

$$\sigma(\xi_{j_1})\, p^*_{\sigma(\xi_{j_1})}(x)>0, \qquad x \in \bar{O}(\xi_{j_1}, \rho)\backslash\{\xi_{j_1}\},$$

we get

$$M(\xi_{j_1}+\sigma_{j_1}\rho)>0.$$

So the continuity of the function implies that there exists an $\bar{\eta}_1 \in Y_{j_1}$ such that

$$M(\bar{\eta}_1)=0. \tag{21}$$

Based on (18), $Q_1(x, \eta_1, ..., \eta_m)$ preserves the same signs for various $x \in \bar{O}(\xi_{j_1}, \rho)$ and $\eta_1 \in \bar{Y}_{j_1}$. So for $x \in \bar{O}(\xi_{j_1}, \rho)$, $Q(x, \eta_1, ..., \eta_m)$ are all strictly monotone increasing (or decreasing) with respect to η_1. And hence $M(\eta_1)$ is strictly monotone as well. So $\bar{\eta}_1 \in \bar{Y}_{j_1}$ satisfying (21) is unique.

Now, we assume inductively that for any $(\eta_v, ..., \eta_m) \in \bar{Y}_{j_v} \times \cdots \times \bar{Y}_{j_m}$, there exists a unique $(\bar{\eta}_1, ..., \bar{\eta}_{v-1}) \in Y_{j_1} \times \cdots \times Y_{j_{v-1}}$ such that

$$M_i(\bar{\eta}_1, ..., \bar{\eta}_{v-1}, \eta_v)=0, \qquad i=1, ..., v-1, \tag{22}$$

where

$$M_i(\bar{\eta}_1, ..., \bar{\eta}_{v-1}, \eta_v):=\min_{x \in \bar{O}(\xi_{j_i}, \rho)} R_i(x, \bar{\eta}_1, ..., \bar{\eta}_{v-1}, \eta_v),$$

$$R_i(x, \bar{\eta}_1, ..., \bar{\eta}_{v-1}, \eta_v):=\sigma(\xi_{j_i})[p^*_{\sigma(\xi_{j_i})}(x)-\varepsilon Q(x, \bar{\eta}_1, ..., \bar{\eta}_{v-1}, \eta_v, ..., \eta_m)].$$

By the arbitrariness of η_v, (22) determines $v-1$ single-valued functions on $\bar{Y}_{j_v}$,

$$\bar{\eta}_i=F_i(\eta_v), \qquad i=1, ..., v-1, \tag{23}$$

with their ranges contained in Y_{j_i}. It can be shown that these functions are continuous. In fact, if there exist an $\bar{\eta}_v \in \bar{Y}_{j_v}$ and an $\{\eta_{vl}\}_{l=1}^{\infty} \subset \bar{Y}_{j_v}$ such that $\eta_{vl} \to \bar{\eta}_v$ $(l \to \infty)$, but $\eta_{il}=F_i(\eta_{vl})$ does not converge to $\bar{\eta}_i=F_i(\bar{\eta}_v)$ for at least one $i \in \{1, ..., v-1\}$, then selecting a subsequence such that $\eta_{il} \to \tilde{\eta}_i$ $(i=1, ..., v-1)$ we have

$$M_i(\tilde{\eta}_1, ..., \tilde{\eta}_{v-1}, \bar{\eta}_v)=0 \quad \text{for} \quad i=1, ..., v-1,$$

and $\tilde{\eta}_i \neq \bar{\eta}_i$ for at least one i, which contradicts the hypothesis of uniqueness.

As with $M(\eta_1)$, when $\eta_v = \xi_{j_v}$ and $\xi_{j_v} + \sigma_{j_v}\rho$, the values of $M_v(\bar{\eta}_1, \ldots, \bar{\eta}_{v-1}, \eta_v)$ (with $\bar{\eta}_i$ subject to (23)) have opposite signs. So by the continuity of M_v we can find $\bar{\eta}_v \in Y_{j_v}$ such that $M_v(\bar{\eta}_1, \ldots, \bar{\eta}_v) = 0$ (where $\bar{\eta}_i = F_i(\bar{\eta}_v)$). Thus we get an $(\bar{\eta}_1, \ldots, \bar{\eta}_v) \in Y_{j_1} \times \cdots \times Y_{j_v}$ for which

$$M_i(\bar{\eta}_1, \ldots, \bar{\eta}_v) = 0, \qquad i = 1, \ldots, v.$$

Moreover, this $(\bar{\eta}_1, \ldots, \bar{\eta}_v)$ is unique. In fact, if there exists another $(\tilde{\eta}_1, \ldots, \tilde{\eta}_v)$ satisfying the above requirements, then we can assume that $|\tilde{\eta}_1 - \bar{\eta}_1| = \max_{i=1,\ldots,v} |\tilde{\eta}_i - \bar{\eta}_i| > 0$. For any $x \in \bar{O}(\xi_{j_1}, \rho)$, by (18) we have

$$|R_1(x, \bar{\eta}_1, \ldots, \bar{\eta}_v) - R_1(x, \tilde{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_v)| > \alpha\, |\tilde{\eta}_1 - \bar{\eta}_1|, \tag{24}$$

and by (19)

$$\begin{aligned} &|R_1(x, \tilde{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_v) - R_1(x, \tilde{\eta}_1, \ldots, \tilde{\eta}_v)| \\ &\quad \leqslant \sum_{i=2}^{v} |R_1(x, \tilde{\eta}_1, \ldots, \tilde{\eta}_{i-1}, \bar{\eta}_i, \ldots, \bar{\eta}_v) - R_1(x, \tilde{\eta}_1, \ldots, \tilde{\eta}_i, \bar{\eta}_{i+1}, \ldots, \bar{\eta}_v)| \\ &\quad \leqslant (v-1)\frac{\alpha}{m}\, |\tilde{\eta}_1 - \bar{\eta}_1| < \alpha\, |\tilde{\eta}_1 - \bar{\eta}_1|. \end{aligned} \tag{25}$$

Suppose $\bar{\xi}, \tilde{\xi} \in \bar{O}(\xi_{j_1}, \rho)$ satisfy

$$R_1(\bar{\xi}, \bar{\eta}_1, \ldots, \bar{\eta}_v) = M_1(\bar{\eta}_1, \ldots, \bar{\eta}_v) = 0,$$
$$R_1(\tilde{\xi}, \tilde{\eta}_1, \ldots, \tilde{\eta}_v) = M_1(\tilde{\eta}_1, \ldots, \tilde{\eta}_v) = 0.$$

Letting $x = \bar{\xi}$ in (24) and (25), we see that $R_1(\bar{\xi}, \tilde{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_v)$ has the same sign as $R_1(\bar{\xi}, \tilde{\eta}_1, \ldots, \tilde{\eta}_v)$. So by $R_1(\bar{\xi}, \tilde{\eta}_1, \ldots, \tilde{\eta}_v) \geqslant 0$ we get

$$R_1(\bar{\xi}, \bar{\eta}_1, \ldots, \bar{\eta}_v) - R_1(\bar{\xi}, \tilde{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_v) < -\alpha\, |\tilde{\eta}_1 - \bar{\eta}_1|.$$

Since (18) implies that Q_1 is sign-preserving for various $x \in \bar{O}(\xi_{j_1}, \rho)$, by (24)

$$R_1(\tilde{\xi}, \bar{\eta}_1, \ldots, \bar{\eta}_v) - R_1(\tilde{\xi}, \tilde{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_v) < -\alpha\, |\tilde{\eta}_1 - \bar{\eta}_1|.$$

Combining this with (25) we get

$$\begin{aligned} R_1(\tilde{\xi}, \bar{\eta}_1, \ldots, \bar{\eta}_v) &= R_1(\tilde{\xi}, \bar{\eta}_1, \ldots, \bar{\eta}_v) - R_1(\tilde{\xi}, \tilde{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_v) \\ &\quad + R_1(\tilde{\xi}, \tilde{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_v) - R_1(\tilde{\xi}, \tilde{\eta}_1, \ldots, \tilde{\eta}_v) < 0, \end{aligned}$$

which contradicts the hypothesis of $M_1(\bar{\eta}_1, \ldots, \bar{\eta}_v) = 0$.

So at last we find $(\bar{\eta}_1, ..., \bar{\eta}_m) \in Y_{j_1} \times \cdots \times Y_{j_m}$ such that

$$\min_{x \in \bar{O}(\xi_{j_i}, \rho)} \sigma(\xi_{j_i})[p^*_{\sigma(\xi_{j_i})}(x) - \varepsilon Q(x, \bar{\eta}_1, ..., \bar{\eta}_m)] = 0, \qquad i = 1, ..., m.$$

If we let $y_{j_i} = \bar{\eta}_i$, q defined by (20) satisfies (15).

The proof is completed.

Acknowledgment

The author is grateful to the referee for his valuable corrections and suggestions, which helped in revising the manuscript.

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